Generating Second Order (Co)homological Information within AT-Model Context *

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Spaniard R+D Project: MTM2016-81030-P: Topological Recognition of 4D digital images via HSF model
Current methodology for Topological Data Analysis

Discrete Object
Cloud of points (CP)

CELLULARIZATION
(mainly, geometrization)

COMPUTATIONAL HOMOLOGICAL ALGEBRA

Equation:
TDA(CP) = Interpretation (CHA (Cell(CP)))
Our long-term motivation

Our Working Hypothesis:
Topology deals with purely non-linear problems
To try to reduce linear algebra work to a minimal expression.
Homology and Homotopy in the same combinatorial classifying scenario
Topological computation= Create maximal “Assymetry” in bitopological dynamics
From a puzzle of small pieces to a puzzle of big pieces (Maximal Homologically Stables Regions)

Our alchemy formula:
TDA(CP) = Interpretation BH(MHSR ((HUGE( TOP(CP))))))
Combinatorial setting: Primal-dual abstract cell complex (pdACC)

- An abstract cell complex $ACC = (E, B, \text{dim})$ is an abstract set $E$, $B$ is an asymmetric, irreflexive and transitive binary relation called the bounding relation among the elements of $E$ and $\text{dim}$ is a function assigning a non-negative integer to each element of $E$ in such a way that if $B(a, b)$, then $\text{dim}(a) < \text{dim}(b)$.


- An primal-dual abstract cell complex $ACC = (E, B_p, B_d, \text{dim})$ is an abstract set $E$, $B_p$ and $B_d$ are asymmetric, irreflexive and transitive binary relations called resp. primal and dual bounding relations among the elements of $E$ and $\text{dim}$ is a function assigning a non-negative integer to each element of $E$ in such a way that if $B_p(a, b)$, then $\text{dim}(a) < \text{dim}(b)$ and $B_d(a, b)$, then $\text{dim}(a) > \text{dim}(b)$.

**Note:** A “geometric” cell complex is in a natural way a primal-dual abstract cell complex being the bounding relation $B_p$ (“to belong to the boundary of”) dual of $B_d$ (“to belong to the coboundary of”). In this context, we can talk about the homology and cohomology derived from $B_p$ and $B_d$, respectively.
Huge scenario for big data (Connectivity graph of cell complexes)

- Working at combinatorial level.
- Incomplete picture within the connectivity graph.
- **Symmetric primal-dual ACC**
- Explosion of the size of the topology computation space

**WHAT FOR??**

Coboundary (dual)

Boundary (primal)

Fig. 1 a) A cell complex $K$ with four 0-cells (1,2,3,4), four 1-cells (a,b,c,d) and one 2-cell (A).
b) The connectivity graph $G(K)$. 
AT-model: exploiting algebraic homology from a representational point of view

Discrete Object

Topological Combinatorial object

\((\mathbb{Z}_2[C], \partial, \phi)\)

AT-MODEL CONTRIBUTION:
FIXED POINT STRATEGY FOR OBTAINING A COMBINATORIAL DESCRIPTION OF THE HOMOLOGY INVERSE OF THE BOUNDARY OPERATOR

\(H(C, \partial) = H(C, \phi)\)

Combinatorial version of AT-model


“Gradient Vector field are not enough for a “correct” combinatorialization of the integral operator of an AT-model”

AT-Segmentation
Crack transport
An AT-model of a torus (coefficient in $\mathbb{Z}_2$)
Primal AT-Rig of a torus

(2,3)-regions: blue
(1,2)-regions: yellow
(0,1)-regions: rojo
Primal AT-Rig of a torus

(2,3)-regions: blue
(1,2)-regions: yellow
(0,1)-regions: rojo
Dual complex and its derived HSF structure
Dual AT-Rig

(2,1)-regions: yellow
(1,0)-regions: red
(0,-1)-regions: green
Dual AT-Rig of a torus

(2,1)-regions: yellow
(1,0)-regions: red
(0,-1)-regions: green
Dual AT-Rig of a torus

(2,1)-regions: yellow
(1,0)-regions: red
(0,-1)-regions: green
AT-RIG of a sphere with two handles

(a) Primal segmentation

(b) Dual segmentation

Fig. 7: AT-segmentations for a sphere with two handles

(a) Primal AT-RIG

(b) Dual AT-RIG

Fig. 8: AT-RIGs for a sphere with two handles
Software-in-progress developed by Helena Molina-Abril
Analysing cup product in cohomology using VOXELO software (2004)

\[
\begin{array}{cccc}
\alpha 1 & \alpha 2 & \alpha 3 & \alpha 4 \\
\alpha 1 & - & CP 1 3 & CP 1 4 \\
\alpha 2 & CP 2 3 & - & \\
\alpha 3 & - & & \\
\alpha 4 & & & \\
\end{array}
\]

CP 1 3: \{ <(721 757 759)>, <(870 871 872)> \}
CP 1 4: \{ <(1112 1130 1131)>, <(1112 1130 1131)>, <(721 757 759)>, <(870 871 872)> \}

\[
\begin{array}{ccccccccc}
\alpha 1 \ast \alpha 2 & \alpha 1 \ast \alpha 3 & \alpha 1 \ast \alpha 4 & \alpha 2 \ast \alpha 3 & \alpha 2 \ast \alpha 4 & \alpha 3 \ast \alpha 4 \\
<(721 757 759)> & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
<(870 871 872)> & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
<(1112 1130 1131)> & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
<(721 757 759)> & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
<(870 871 872)> & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
TORO B
COMPLEX WITH 6676 cells 1184 0-cells, 2898 1-cells,
2154 2-cells 440 3-cells

Software-in-progress developed by Helena Molina-Abril
Analysing cup product in cohomology using VOXELO software (2004)

\[
\begin{array}{c|cccc}
\text{alpha 1} & \text{alpha 2} & \text{alpha 3} & \text{alpha 4} \\
\hline
\text{alpha 1} & CP 1 & 2 & - & - \\
\text{alpha 2} & - & CP 2 & 3 & CP 2 & 4 \\
\text{alpha 3} & - & - & CP 5 & 2 & - \\
\text{alpha 4} & - & - & - & CP 6 & 2 & - \\
\end{array}
\]

\[
\text{CP 1 2: } \{ <(495 521 522)> <(1110 1111 1112)> \}
\text{CP 2 3: } \{ <(1110 1111 1112)> <(1177 1183 1184)> \}
\text{CP 2 4: } \{ <(1110 1111 1112)> \}
\]

\[
\begin{array}{cccccccc}
\text{alpha 1} & * & \text{alpha 2} & \text{alpha 1} & * & \text{alpha 3} & \text{alpha 1} & * & \text{alpha 4} \\
\hline
<(495 521 522)> & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
<(1110 1111 1112)> & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
<(1110 1111 1112)> & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
<(1177 1183 1184)> & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
<(1110 1111 1112)> & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]
Conclusions

- AT-RIG is a topological feature that can help to discriminate at homotopy level objects that are homologically equivalent (same Betti numbers).

- In a near future, we deal with the problems for promoting AT-RIG to topological invariant (both in the cases of depending or not of a ground ring).
¡Gracias!